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# NUMERICAL AND ASYMPTOTIC SOLUTION OF THE EQUATIONS OF PROPAGATION OF HYDROELASTIC VIBRATIONS IN A CURVED PIPE

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*A mathematical model for propagation of hydroelastic waves in a pipe is developed using the equations of motion of a shell and a fluid. A method for deriving two-dimensional equations is proposed, and asymptotic formulas for solutions of these equations are obtained. A model problem is solved numerically, and the results are compared with data obtained by others. The results obtained make it possible to calculate the propagation of pressure waves for an arbitrary (within the framework of the assumptions made) shape of the axial line of the pipe and can be used in designing systems for diagnostics of pipeline performance.*

The motion of a fluid in pipes is a classical problem of mechanics. Recently, phenomena associated with instability of a long elastic pipe with a fluid flow have attracted a considerable amount of interest. Consequences of the instability can be displacement of an underground pipe from its initial position and rising of a segment of an underwater pipeline to the surface.

Distortion of the pipeline profile must be detected as soon as possible. This can be done by analyzing a pressure pulse (hydraulic jump) or an acoustic wave that passed through the fluid flow. For passage of a wave through the pipeline, the time dependence of the pressure is determined by the curvature of the pipeline axis.

Hydroelastic vibrations of a curved pipeline occur under the action of the elastic properties of the pipe wall, the pressure and friction in the fluid flow, and the resistance of the ambient medium. The propagation of the vibrations allowing for these factors has not been adequately studied.

One-dimensional mathematical models of unsteady fluid flow in pipes were considered in [1, 2]. An analysis of the theories of a hydraulic jump is given in [3]. These models, however, do not allow us to study the effect of the profile bend on pressure-wave propagation.

Vol'mir [4] and other researchers studied the interaction of cylindrical shells with fluid flows using the equations of general hydroelasticity theory. However, this approach is efficient only for pipes of medium length. There are also models for flow of a viscous incompressible fluid in curved pipes that were developed for investigation of blood flows (see, e.g., [5, 6]). The dynamics of these systems is considerably different from the pipeline dynamics, and the results obtained in these papers are inapplicable to the problem considered herein.

A hydraulic jump propagating through a pipeline moving according to a specified law was considered by Yaskelyain [7], who developed a one-dimensional mathematical model and performed numerical calculations using the method of characteristics. A one-dimensional mathematical model of a hydraulic jump in a curved pipeline was proposed by Ovchinnikov [8] (see also the review of literature in [9]).

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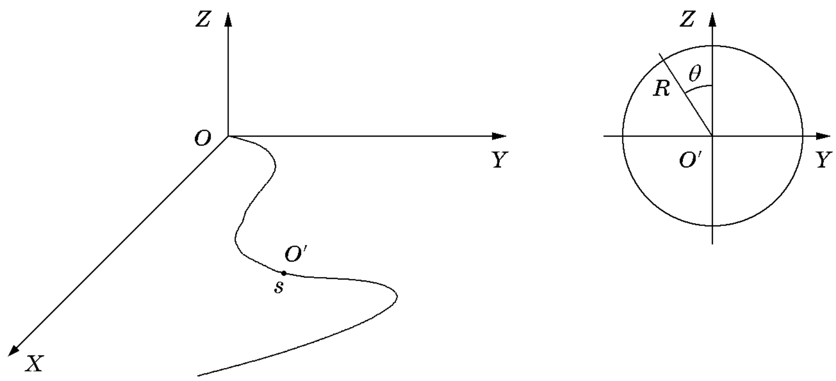


Fig. 1

All the above-mentioned mathematical models are inapplicable for the analysis of the propagation of pressure oscillations through a curved underground pipeline because the motion of a pressure wave through a pipeline must be determined taking into account phenomena with a characteristic length scale of the order of the pipe radius. It is also necessary to take into account the interaction of the pipe wall with both the ambient medium and the fluid flow.

In the present paper, we consider the propagation of a pressure wave in a fluid flow in a curved elastic pipeline taking into account the effect of the ambient medium. This problem is related to the problem of diagnostics of distortion of an underground pipeline profile under the action of internal fluid flow. If a pipeline is probed by acoustic waves, the pressure at each point of the pipeline as a function of time will depend on the curvature of the axial line. The aim of this paper is to construct and test a mathematical model that would allow us to study this dependence.

**1. Physical Formulation of the Problem.** Let a fluid flow uniformly inside a curved underground pipeline whose axis is a two-dimensional curve $\Gamma = \{(x, y)\colon\ x = x(s),\ y = y(s);\ 0 \leqslant s \leqslant L\}$. At the initial point of the pipeline $s = 0$, a periodic force is applied to the fluid flow and generates pressure oscillations in the flow. The problem is to study the dynamics of the system.

In constructing a mathematical model, we assume that the fluid motion can be linearized in the neighborhood of the steady flow, the effect of the ground can be taken into account via boundary conditions, and the following parameters are small: $\alpha = R_0/l$ ($R_0$ is the pipe radius and $l$ is the minimum wavelength of the signal), $\varepsilon = R_0/\min|\rho_0|$ ($\rho_0$ is the radius of curvature of the pipeline axis), and $h^* = h/R_0$ ($h$ is the thickness of the pipe wall). The motion of the wall is described by linear shell theory. Friction in the fluid in oscillatory processes is ignored.

We introduce the following curvilinear orthogonal coordinates: $s$ is the distance along the pipe axis and $\theta$ and $R$ are the angle and radius of polar coordinates in the cross section at point $s$ (Fig. 1). The Cartesian coordinates of the point are given by

$$X = x(s) + \frac{dy(s)}{ds} R\sin\theta, \quad Y = y(s) - \frac{dx(s)}{ds} R\sin\theta, \quad Z = R\cos\theta. \tag{1}$$

Similar coordinates were used in [10], where the pipe had the shape of a torus and the angle $\varphi$ was taken as the coordinate $s$. Following [11], from (1) we can determine the components of the metric tensor, the Christoffel symbols, and the Lamé coefficients for the orthogonal coordinate system constructed. The first Lamé coefficient can be written as

$$\sqrt{g_{11}} = 1 + \frac{R}{\rho_0(s)} \sin\theta.$$

Simultaneous motion of a fluid and a pipe was considered in [12], where the problem had only been formulated. In the present paper, we refine the method of linearization of the equations of motion for a fluid and focus on the study of a three-dimensional mathematical model.

**2. Three-Dimensional Initial-Boundary-Value Problem of Motion of the System.** The motion of a pipeline is governed by the equations of an elastic body [11]

$$\rho_t a^k = \nabla_i p^{ki}, \tag{2}$$

where $\rho_t$ is the density of the pipe material, $a^k$ are the acceleration components, $p^{ki}$ are the stress-tensor components, and $\nabla_i$ is the covariant derivative.

It is assumed that the normal fluid pressure force and the force of entrainment of the wall by the steady flow act on the internal surface of the pipe, and the external surface of the pipe is acted upon by the normal ambient pressure, the force of elastic resistance of the ambient medium to the radial displacement of the wall, and the friction force, proportional to the velocity of tangential motion of the wall. Then, for the density of the surface forces, we obtain

$$\begin{gathered} \boldsymbol{P}_n\Big|_{R=R_0-h/2} = \Phi(v_{s0})\cdot \boldsymbol{e}_s + p\cdot \boldsymbol{e}_R, \\ \boldsymbol{P}_n\Big|_{R=R_0+h/2} = -kp_e\Big(\frac{\partial w_s}{\partial t}\,\boldsymbol{e}_s + \frac{\partial w_\theta}{\partial t}\,\boldsymbol{e}_\theta\Big) - (p_e + æ w_R)\boldsymbol{e}_R. \end{gathered} \tag{3}$$

Here $\boldsymbol{e}_s$, $\boldsymbol{e}_\theta$, and $\boldsymbol{e}_R$ are the unit basis vectors, $w_s$, $w_\theta$, and $w_R$ are the physical components of the displacement, $k$ and $æ$ are the friction coefficient and the elasticity of the ambient medium, $p$ and $p_e$ are the pressures of the fluid and the medium, and $\Phi(v_{s0})$ is a function that describes the force of friction between the flow and the wall.

Now, using (2) and (3) and passing to the equations of technical moment shell [13], we obtain the boundary-value problem

$$\begin{gathered}
\frac{\alpha}{A}\,\frac{\partial I'}{\partial\zeta} - (1-\nu)\,\frac{\partial\chi'}{\partial\theta} + \frac{1-\nu}{A}\Big(\varepsilon f u'\sin\theta - \alpha\,\frac{\partial w'}{\partial\zeta}\Big) = -\frac{1-\nu^2}{Eh^*}F_s, \\
\frac{\partial I'}{\partial\theta} + (1-\nu)\,\frac{\alpha}{A}\,\frac{\partial\chi'}{\partial\zeta} + (1-\nu)\,\frac{\varepsilon f}{A}\,\sin\theta\Big(v' - \frac{\partial w'}{\partial\theta}\Big) = -\frac{1-\nu^2}{Eh^*}\,F_\theta, \\
-\Big(1 + \frac{\varepsilon f}{A}\,\sin\theta\Big)I' + \frac{1-\nu}{A}\Big[2\varepsilon f w'\sin\theta + \alpha\,\frac{\partial u'}{\partial\zeta} + \varepsilon f\,\frac{\partial}{\partial\theta}\,(v'\sin\theta)\Big] \\
-\frac{(h^*)^2}{12}\tilde\nabla^2 w' - \frac{(h^*)^2}{12}\,\tilde\nabla^2\tilde\nabla^2 w' = -\frac{1-\nu^2}{Eh^*}\,F_R, \\
I' = \frac{1}{A}\Big(\alpha\,\frac{\partial u'}{\partial\zeta} + \varepsilon f v'\cos\theta\Big) + \frac{\partial v'}{\partial\theta} + \Big(1 + \frac{R_0\sin\theta}{\rho_0(\zeta)A}\Big)w', \\
\chi' = \frac{1}{2A}\Big(\alpha\,\frac{\partial v'}{\partial\zeta} - \varepsilon f u'\cos\theta\Big) - \frac{1}{2}\,\frac{\partial u'}{\partial\theta}, \quad \tilde\nabla^2 w' = \frac{1}{A}\Big[\alpha^2\,\frac{\partial}{\partial\zeta}\Big(\frac{1}{A}\,\frac{\partial w'}{\partial\zeta}\Big) + \frac{\partial}{\partial\theta}\Big(A\,\frac{\partial w'}{\partial\theta}\Big)\Big], \\
\frac{1}{h^*}\,F_s = -\rho_t R_0^2\omega^2\,\frac{\partial^2 u'}{\partial\tau^2} + \frac{(h^*)^2}{12}\Big(1 + 2\,\frac{\varepsilon f}{A}\,\sin\theta\Big)\frac{\alpha}{A}\,\rho_t R_0^2\omega^2\,\frac{\partial^2}{\partial\tau^2}\,\frac{\partial w'}{\partial\zeta} \\
+\frac{1}{h^*}\,\Big[\Phi_t(v_{s0}) - kp_e\omega R_0\,\frac{\partial}{\partial\tau}\Big(u' - \frac{h^*\alpha}{2A}\,\frac{\partial w'}{\partial\zeta}\Big)\Big],
\end{gathered} \tag{4}$$

$$\frac{1}{h^*}\,F_\theta = -\rho_t R_0^2\omega^2\,\frac{\partial^2 v'}{\partial\tau^2} + \frac{(h^*)^2}{12}\Big(2 + \frac{\varepsilon f}{A}\,\sin\theta\Big)\rho_t R_0^2\omega^2\,\frac{\partial^2}{\partial\tau^2}\,\frac{\partial w'}{\partial\theta} - \frac{1}{h^*}\,kp_e R_0\,\frac{\partial}{\partial\tau}\Big(v' - \frac{h^*}{2}\,\frac{\partial w'}{\partial\theta}\Big),$$

$$\frac{1}{h^*} F_R = -\rho_t R_0^2 \omega^2 \frac{\partial^2 w'}{\partial \tau^2} + \frac{(h^*)^2}{12} \rho_t R_0^2 \omega^2 \frac{\partial^2}{\partial \tau^2} \Big[ -\Big(2 + \frac{\varepsilon f}{A} \sin\theta\Big) \frac{\partial v'}{\partial \theta}$$

$$-\frac{\alpha}{A}\Big(1 + 2\frac{\varepsilon f}{A} \sin\theta\Big) \frac{\partial u'}{\partial \zeta} + \frac{\varepsilon f}{A} \cos\theta \left(\frac{\partial w'}{\partial \theta} - 3v'\right) - \frac{\alpha}{A} \frac{\varepsilon f'}{A^2} \sin\theta \Big(2u' + \alpha \frac{\partial w'}{\partial \zeta}\Big) + \frac{\partial^2 w'}{\partial \theta^2}$$

$$+ \frac{\alpha^2}{A^2} \frac{\partial^2 w'}{\partial \zeta^2}\Big] + \frac{1}{h^*} (p - p_e - æ R_0 w') - \frac{1}{2} \Big\{ \frac{\varepsilon f}{A} k p_e \omega R_0 \cos\theta \frac{\partial}{\partial \tau} \Big( v' - \frac{h^*}{2} \frac{\partial w'}{\partial \theta} \Big)$$

$$+ k\omega R_0 \frac{\partial}{\partial \theta} \Big[ p_e \frac{\partial}{\partial \tau} \Big( v' - \frac{h^*}{2} \frac{\partial w'}{\partial \theta} \Big) \Big] + \frac{\alpha}{A} k p_e \omega R_0 \frac{\partial^2}{\partial \tau \, \partial \zeta} \Big( u' - \frac{h^* \alpha}{2A} \frac{\partial w'}{\partial \zeta} \Big) \Big\},$$

$$A = 1 + \frac{R_0}{\rho_0(s)} \sin\theta, \quad f = \frac{\min_{0<\zeta<L} |\rho_0(\zeta)|}{\rho_0(\zeta)}, \qquad w' = v' = u' = 0, \quad \frac{\partial w'}{\partial \zeta} = 0 \quad \text{for} \quad \zeta = 0, L.$$

Here $u'$, $v'$, and $w'$ are the component of the displacement of the middle surface of the pipe wall normalized by $R_0$, $\tau = \omega t$ and $\zeta = s/l$ are the dimensionless time and longitudinal coordinate, $\nu$ and $E$ are Poisson's ratio and Young's modulus for the pipe, $\rho_0(\zeta)$ is the radius of curvature of the axis of the pipeline, and $F_s$, $F_\theta$, and $F_R$ are the densities of the forces acting in the corresponding directions. System (4) must be supplemented with boundary conditions. We obtain them by solving the equations of equilibrium of the pipeline with steady fluid flow that are obtained from (4) if the time derivatives and unsteady characteristics of the fluid are set equal to zero.

The motion of the fluid is described by the Euler equations [11] with a friction term on the right side:

$$\rho\Big(\frac{\partial \boldsymbol{v}}{t} + (\boldsymbol{v}, \nabla)\, \boldsymbol{v}\Big) = -\operatorname{grad} p - \boldsymbol{\Phi}(v_{s0}), \qquad \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \boldsymbol{v}) = 0.$$

Following [12], we introduce the representations of solutions

$$\boldsymbol{v} = \boldsymbol{v}_0 + \boldsymbol{v}_1, \qquad \rho = \rho_f + \rho_1, \qquad p = p_0 + p_1,$$

where $\boldsymbol{v}_0$ and $p_0$ is the solution of the equations

$$(\boldsymbol{v}_0, \nabla)\boldsymbol{v}_0 = -\frac{1}{\rho_f} \operatorname{grad} p_0 - \frac{1}{\rho_f} \boldsymbol{\Phi}(v_{s0}), \qquad \operatorname{div} \boldsymbol{v}_0 = 0, \qquad \rho_f = \text{const},$$

Omitting terms that are nonlinear in $\boldsymbol{v}_1$, $\rho_1$, and $p_1$ and taking into account the equation of state for the density perturbations $\rho_1 = p_1/c_f^2$, we obtain the following unsteady linearized equations of motion of the fluid:

$$\rho_f\Big(\frac{\partial \boldsymbol{v}_1}{\partial t} + (\boldsymbol{v}_0, \nabla)\boldsymbol{v}_1 + \Big[(\boldsymbol{v}_1, \nabla)\boldsymbol{v}_0\Big]_c\Big) = -\operatorname{grad} p_1, \quad \frac{1}{c_f^2}\Big[\frac{\partial p_1}{\partial t} + \operatorname{div}(p_1 \boldsymbol{v}_0)\Big] + \operatorname{div}(\rho_f \boldsymbol{v}_1) = 0,$$

where $c_f$ is the speed of sound in the fluid. We assume that $\boldsymbol{v}_0$ depends weakly on the coordinate, and, therefore, only the part $(\boldsymbol{v}_1, \nabla)\boldsymbol{v}_0$ of the term $[(\boldsymbol{v}_1, \nabla)\boldsymbol{v}_0]_c$, which does not contain partial derivatives, remains in the equations.

We introduce the dimensionless functions

$$\boldsymbol{v}' = \frac{\boldsymbol{v}_1}{\omega l}, \qquad \boldsymbol{v}_0' = \frac{\boldsymbol{v}_0}{\omega l}, \qquad p' = \frac{p_1}{p_a}, \qquad p_0' = \frac{p_0}{p_a}$$

($p_a$ is the atmospheric pressure) and denote their components as

$$\boldsymbol{v}' = (v_s, v_\theta, v_r), \qquad \boldsymbol{v}_0' = (v_{s0}, v_{\theta 0}, v_{r0}).$$

Then, $a^2 = p_a/(\rho_f \omega^2 l^2)$ is a dimensionless parameter. For a steady motion, we obtain

$$\frac{v_{s0}}{\sqrt{g_{11}}}\frac{\partial v_{s0}}{\partial \zeta} + \frac{v_{\theta 0}}{\alpha r}\frac{\partial v_{s0}}{\partial \theta} + \frac{v_{r0}}{\alpha}\frac{\partial v_{s0}}{\partial r} + \frac{\varepsilon}{\alpha}\frac{f(\zeta)}{\sqrt{g_{11}}} v_{s0}(v_{\theta 0}\cos\theta + v_{r0}\sin\theta) = -\frac{a^2}{\sqrt{g_{11}}}\frac{\partial p_0'}{\partial \zeta} - \frac{l}{\rho_f}\Big|\mathbf{\Phi}(v_{s0})\Big|,$$

$$\frac{v_{s0}}{\sqrt{g_{11}}}\frac{\partial v_{\theta 0}}{\partial \zeta} + \frac{v_{\theta 0}}{\alpha r}\frac{\partial v_{\theta 0}}{\partial \theta} + \frac{v_{r0}}{\alpha}\frac{\partial v_{\theta 0}}{\partial r} - \frac{\varepsilon}{\alpha}\frac{f(\zeta)}{\sqrt{g_{11}}} v_{s0}^2\cos\theta + \frac{v_{r0}v_{\theta 0}}{\alpha r} = -\frac{a^2}{\alpha r}\frac{\partial p_0'}{\partial \theta},$$

$$\frac{v_{s0}}{\sqrt{g_{11}}}\frac{\partial v_{r0}}{\partial \zeta} + \frac{v_{\theta 0}}{\alpha r}\frac{\partial v_{r0}}{\partial \theta} + \frac{v_{r0}}{\alpha}\frac{\partial v_{r0}}{\partial r} - \frac{v_{\theta 0}^2}{\alpha r} - \frac{\varepsilon}{\alpha}\frac{f(\zeta)}{\sqrt{g_{11}}} v_{s0}^2\sin\theta = -\frac{a^2}{\alpha}\frac{\partial p_0'}{\partial r}, \tag{5}$$

$$\frac{1}{\sqrt{g_{11}}}\frac{\partial v_{s0}}{\partial \zeta} + \frac{1}{\alpha r}\frac{\partial v_{\theta 0}}{\partial \theta} + \frac{1}{\alpha}\frac{\partial v_{r0}}{\partial r} + \frac{v_{r0}}{\alpha r} + \frac{\varepsilon}{\alpha}\frac{f(\zeta)}{\sqrt{g_{11}}}(v_{\theta 0}\cos\theta + v_{r0}\sin\theta) = 0,$$

$$v_{s0}(0,\theta,r) = v_0, \qquad v_{r0}(\zeta,\theta,1) = 0, \qquad p_0'(L,\theta,r) = 1.$$

For unsteady motion, we have

$$\frac{\partial v_s}{\partial \tau} + \frac{v_{s0}}{\sqrt{g_{11}}}\frac{\partial v_s}{\partial \zeta} + \frac{v_{\theta 0}}{\alpha r}\frac{\partial v_s}{\partial \theta} + \frac{v_{r0}}{\alpha}\frac{\partial v_s}{\partial r} + v_{s0}\frac{\varepsilon f(\zeta)}{\alpha\sqrt{g_{11}}}(v_\theta\cos\theta + v_r\sin\theta)$$

$$+ v_s\frac{\varepsilon f(\zeta)}{\alpha\sqrt{g_{11}}}(v_{\theta 0}\cos\theta + v_{r0}\sin\theta) = -\frac{a^2}{\sqrt{g_{11}}}\frac{\partial p'}{\partial \zeta},$$

$$\frac{\partial v_\theta}{\partial \tau} + \frac{v_{s0}}{\sqrt{g_{11}}}\frac{\partial v_\theta}{\partial \zeta} + \frac{v_{\theta 0}}{\alpha r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_{r0}}{\alpha}\frac{\partial v_\theta}{\partial r} - 2v_{s0}v_s\frac{\varepsilon f(\zeta)}{\alpha\sqrt{g_{11}}}\cos\theta + \frac{v_\theta v_{r0}}{\alpha r} + \frac{v_{\theta 0}v_r}{\alpha r} = -\frac{a^2}{\alpha r}\frac{\partial p'}{\partial \theta},$$

$$\frac{\partial v_r}{\partial \tau} + \frac{v_{s0}}{\sqrt{g_{11}}}\frac{\partial v_r}{\partial \zeta} + \frac{v_{\theta 0}}{\alpha r}\frac{\partial v_r}{\partial \theta} + \frac{v_{r0}}{\alpha}\frac{\partial v_r}{\partial r} - 2v_{s0}v_s\frac{\varepsilon f(\zeta)}{\alpha\sqrt{g_{11}}}\sin\theta - \frac{v_{\theta 0}v_\theta}{\alpha r} = -\frac{a^2}{\alpha}\frac{\partial p'}{\partial r}, \tag{6}$$

$$a^2\Big(\frac{\partial p'}{\partial \tau} + \frac{v_{s0}}{\sqrt{g_{11}}}\frac{\partial p'}{\partial \zeta} + \frac{v_{\theta 0}}{\alpha r}\frac{\partial p'}{\partial \theta} + \frac{v_{r0}}{\alpha}\frac{\partial p'}{\partial r}\Big) + \frac{1}{\sqrt{g_{11}}}\frac{\partial v_s}{\partial \zeta} + \frac{1}{\alpha r}\frac{\partial v_\theta}{\partial \theta}$$

$$+ \frac{1}{\alpha}\frac{\partial v_r}{\partial r} + \frac{v_r}{\alpha r} + \frac{\varepsilon f(\zeta)}{\alpha\sqrt{g_{11}}}(v_\theta\cos\theta + v_r\sin\theta) = 0,$$

$$p'(0,\theta,r,\tau) = F_0(\tau), \quad p(L,\theta,r,\tau) = 0, \quad v_r(\zeta,\theta,1,\tau) = \alpha\frac{\partial w'}{\partial \tau}(\zeta,\theta,\tau).$$

Here the equations for steady motion are supplemented with the condition of no normal flow through the wall. In addition, we specify constant velocity at the pipeline entrance and impose the condition of pressure equalization at the end of the pipeline. For unsteady motion, we specify the pressure at the entrance and impose the condition of no normal flow through the wall and the condition of pressure equalization at the end of the pipeline.

The resistance to a steady flow is prescribed in the form [2, 14]

$$\Big|\mathbf{\Phi}(v_{s0})\Big| = \beta v_0^2, \quad \beta = \frac{\lambda\rho_f}{4R_0}, \quad \lambda = \begin{cases} 64/\mathrm{Re}, & \mathrm{Re} < 2000, \\ 0.0032 + 0.221/\mathrm{Re}^{0.237}, & \mathrm{Re} \geqslant 2000, \end{cases} \quad \mathrm{Re} = \frac{2v_0R_0}{\mu},$$

where $\mu$ is the kinematic viscosity. At the initial time, the values of the unknown functions in (6) are taken to be zero.

Thus, we have formulated the initial-boundary-value problem (4)–(6) of simultaneous motion of the pipeline wall and the fluid.

**3. Reduction of the Problem to a Two-Dimensional Problem and Obtaining a Steady Solution.** To eliminate the angular variable $\theta$, we use the expansion obtained in [12]. For the pipeline, it is given by

$$\begin{aligned} u' &= u^{(0)}(\zeta,\tau) + \varepsilon u^{(1)}(\zeta,\tau)\sin\theta + \varepsilon u^{(2)}(\zeta,\tau)\cos\theta + O(\varepsilon^2),\\ v' &= v^{(0)}(\zeta,\tau) + \varepsilon v^{(1)}(\zeta,\tau)\sin\theta + \varepsilon v^{(2)}(\zeta,\tau)\cos\theta + O(\varepsilon^2),\\ w' &= w^{(0)}(\zeta,\tau) + \varepsilon w^{(1)}(\zeta,\tau)\sin\theta + \varepsilon w^{(2)}(\zeta,\tau)\cos\theta + O(\varepsilon^2). \end{aligned} \tag{7}$$

For the fluid ($k = s, \theta, r$), it has the form

$$\begin{aligned} v_k(\tau,\zeta,\theta,r) &= v_k^{(0)}(\tau,\zeta,r) + \varepsilon v_k^{(1)}(\tau,\zeta,r)\sin\theta + \varepsilon v_k^{(2)}(\tau,\zeta,r)\cos\theta + O(\varepsilon^2),\\ p'(\tau,\zeta,\theta,r) &= p^{(0)}(\tau,\zeta,r) + \varepsilon p^{(1)}(\tau,\zeta,r)\sin\theta + \varepsilon p^{(2)}(\tau,\zeta,r)\cos\theta + O(\varepsilon^2). \end{aligned} \tag{8}$$

This representation is due to the fact that, after expanding the solution of problem (4) in a power series in the small parameter $\varepsilon$ and then in a Fourier series in $\theta$ (since the periodicity in $\theta$ is obvious), all Fourier coefficients (except for the first) are equal to zero in a zeroth approximation because they are solutions of boundary-value problems with zero right sides and homogeneous boundary conditions. In a first approximation for $\varepsilon$, all Fourier coefficients are equal to zero, except for the coefficients at $\sin\theta$, $\cos\theta$, etc.

For steady motion of the fluid in a zeroth approximation for $\varepsilon$, we obtain

$$v_{s0}^{(0)} = v_0, \quad v_{\theta 0}^{(0)} = v_{r0}^{(0)} = 0, \quad p_0^{(0)} = 1 + \frac{l\beta}{\rho_f a^2}\, v_0^2 (L - \zeta). \tag{9}$$

As a first approximation (expanding the unknown functions in an asymptotic series in $\alpha$ in a long-wave approximation), we obtain

$$\begin{aligned} p_0^{(1)} &= r\,\frac{f(\zeta)}{a^2}\, v_0^2 + \alpha^2\, \frac{r}{8}\,(3 - r^2)\,\frac{v_0^2}{a^2}\Big(\frac{\partial^2 f}{\partial\zeta^2} + \frac{l\beta}{\rho_f}\,\frac{\partial f}{\partial\zeta}\Big),\\ v_{s0}^{(1)} &= -rfv_0 - \alpha^2\,\frac{rv_0}{8}\,(3 - r^2)\Big(\frac{\partial^2 f}{\partial\zeta^2} + \frac{l\beta}{\rho_f}\,\frac{\partial f}{\partial\zeta}\Big) - r\,\frac{l\beta}{\rho_f}\, v_0 \int_0^\zeta f\, d\zeta,\\ v_{r0}^{(1)} &= -\frac{3}{8}\,\alpha v_0 (1 - r^2)\Big(\frac{\partial f}{\partial\zeta} + \frac{l\beta}{\rho_f}\, f\Big), \quad v_{\theta 0}^{(2)} = -\frac{\alpha v_0}{8}\,(3 - r^2)\Big(\frac{\partial f}{\partial\zeta} + \frac{l\beta}{\rho_f}\, f\Big). \end{aligned} \tag{10}$$

In the leading order, for $r = 1$ the quantity $p_0^{(1)}$ is equal to

$$p_0^{(1)} \approx \frac{f(\zeta)}{a^2}\, v_0^2. \tag{11}$$

From the physical viewpoint, this quantity corresponds to the pressure difference between $\theta_1 = \pi/2$ and $\theta_2 = -\pi/2$, i.e., at the distance equal to the diameter of the pipeline (the so-called effect of centrifuge, see, e.g., [6]).

Substitution of (7) into the boundary-value problem (4) yields one-dimensional equations for the expansion coefficients. Setting the time derivatives in these equations equal to zero, we obtain the boundary-value problem for the equilibrium of the pipeline, whose solution is the initial condition for the one-dimensional, initial-boundary-value problem for the expansion coefficients.

In a zeroth approximation for $\varepsilon$, the equilibrium problem can be solved exactly:

$$u_0^{(0)} = b_1 + b_2\zeta - \frac{\nu}{\alpha}\int \bar w\, d\zeta + c\zeta^2, \qquad \bar w = w_0^{(0)} - d - g\zeta,$$

$$\begin{aligned} w_0^{(0)} = \Big[A_1 \cos\Big(\delta_2\,\frac{\zeta}{\alpha}\Big) + A_2 \sin\Big(\delta_2\,\frac{\zeta}{\alpha}\Big)\Big] \exp\Big(\delta_1\,\frac{\zeta}{\alpha}\Big) + \Big[A_3 \cos\Big(\delta_2\,\frac{\zeta}{\alpha}\Big)\\ + A_4 \sin\Big(\delta_2\frac{\zeta}{\alpha}\Big)\Big] \exp\Big(-\delta_1\,\frac{\zeta}{\alpha}\Big) - \frac{\alpha\nu b_2}{1 + K_{\rm g} - \nu^2} + d + g\zeta, \end{aligned}$$

where $K_{\rm g} = R_0 æ/(Eh^*)$ is the elasticity of the ground, the coefficients $c$, $d$, $g$, $\delta_1$, and $\delta_2$ are obtained from the right sides, and $b_1$, $b_2$, $A_1$, $A_2$, $A_3$, and $A_4$ are obtained from the boundary conditions.

To automate the calculations, we wrote a code using the programming language Reduce. We do not present the analytical expressions obtained because they are too cumbersome.

For boundary-value problems describing the equilibrium of the pipeline in a first approximation for $\varepsilon$, we constructed finite-difference schemes by an integrointerpolation method. We wrote a computer program that solves the system of algebraic equations obtained using the method of minimum residuals.

Thus, the steady flow of the fluid and the equilibrium position of the pipeline are determined with accuracy up to quantities of the order of $\varepsilon\alpha^2$.

**4. Solution of the Unsteady Problem.** Substituting (7) and (8) into Eqs. (4) and (6) and equating the coefficients at $\varepsilon^0$, $\varepsilon\sin\theta$, and $\varepsilon\cos\theta$ on the left and right sides of the equations and the boundary conditions, we obtain a two-dimensional, initial-boundary-value problem for the coefficients of expansions (7) and (8). Expanding the solutions of the obtained fluid equations in a power series in the small parameter $\alpha$, in the long-wave approximation we obtain

$$\frac{\partial v_{s0}}{\partial\tau} + v_0\,\frac{\partial v_{s0}}{\partial\zeta} = -a^2\,\frac{\partial p_0}{\partial\zeta}, \quad a^2\Big(\frac{\partial p_0}{\partial\tau} + v_0\,\frac{\partial p_0}{\partial\zeta}\Big) + \frac{\partial v_{s0}}{\partial\zeta} + 2\,\frac{\partial w^{(0)}}{\partial\tau} = 0,$$

$$p_0\Big|_{\zeta=0} = F_0(\tau), \quad p_0\Big|_{\zeta=L} = 0, \quad v_{s0}\Big|_{\tau=0} = p_0\Big|_{\tau=0} = 0, \qquad (12)$$

$$v_s^{(0)} = v_{s0} + \alpha^2\,\frac{r^2}{2}\,\frac{\partial^2 w^{(0)}}{\partial\tau\,\partial\zeta}, \quad v_r^{(0)} = \alpha r\,\frac{\partial w^{(0)}}{\partial\tau}, \quad p^{(0)} = p_0 - \frac{\alpha^2 r^2}{2a^2}\,\frac{\partial}{\partial\tau}\Big(\frac{\partial w^{(0)}}{\partial\tau} + v_0\,\frac{\partial w^{(0)}}{\partial\zeta}\Big).$$

As a first approximation for $\varepsilon$, using similar expansions in $\alpha$, we obtain the asymptotic formulas

$$p^{(1)} = \frac{2rv_0}{a^2}\,fv_{s0} + \alpha^2 r\,\frac{v_0}{a^2}\Big[\Big(\frac{5}{2}\,f + \frac{l\beta}{\rho_f}\int\limits_0^\zeta f\,d\zeta\Big)\frac{\partial^2 w^{(0)}}{\partial\tau\,\partial\zeta}$$

$$-\frac{1}{v_0}\,\frac{\partial}{\partial\tau}\Big(\frac{\partial w^{(1)}}{\partial\tau} + v_0\,\frac{\partial w^{(1)}}{\partial\zeta}\Big) + \frac{1}{8}\,(r^2-3)F(\tau,\zeta)\Big],$$

$$p^{(2)} = -\frac{\alpha^2 r}{a^2}\,\frac{\partial}{\partial\tau}\Big(\frac{\partial w^{(2)}}{\partial\tau} + v_0\,\frac{\partial w^{(2)}}{\partial\zeta}\Big), \qquad (13)$$

$$F(\tau,\zeta) = \Big(\frac{\partial}{\partial\tau} + v_0\,\frac{\partial}{\partial\zeta}\Big)^2(fv_{s0}) - \frac{\partial^2(fv_{s0})}{\partial\zeta^2} + \frac{\partial}{\partial\zeta}\Big[\Big(2f + \frac{l\beta}{\rho_f}\int\limits_0^\zeta f d\zeta\Big)\frac{\partial v_{s0}}{\partial\zeta}\Big]$$

$$+\frac{a^2}{v_0}\,\frac{\partial}{\partial\zeta}\Big(f\,\frac{\partial p_0}{\partial\zeta}\Big) + 3\Big(2f + \frac{l\beta}{\rho_f}\int\limits_0^\zeta f\,d\zeta\Big)\frac{\partial^2 w^{(0)}}{\partial\tau\,\partial\zeta} - \Big(2f' + \frac{l\beta}{\rho_f}\,f\Big)\frac{\partial w^{(0)}}{\partial\tau}$$

$$-\Big(\frac{\partial}{\partial\tau} + v_0\,\frac{\partial}{\partial\zeta}\Big)\Big[a^2\Big(2f + \frac{l\beta}{\rho_f}\int\limits_0^\zeta f\,d\zeta\Big)\frac{\partial p_0}{\partial\zeta} + \frac{f}{v_0}\Big(\frac{\partial v_{s0}}{\partial\zeta} - \frac{\partial w^{(0)}}{\partial\tau}\Big)\Big].$$

The functions $w^{(0)}$, $w^{(1)}$, and $w^{(2)}$ in (12) and (13) are components of expansion (7) for the radial displacement of the pipe wall. The boundary-value problem for $w^{(0)}$ is solved simultaneously with (12) since it contains $p^{(0)}$. The corresponding quantities $w^{(1)}$ and $w^{(2)}$ appearing in the equations for $p^{(1)}$ and $p^{(2)}$ can be eliminated using Eqs. (13). Therefore, they are determined independently.

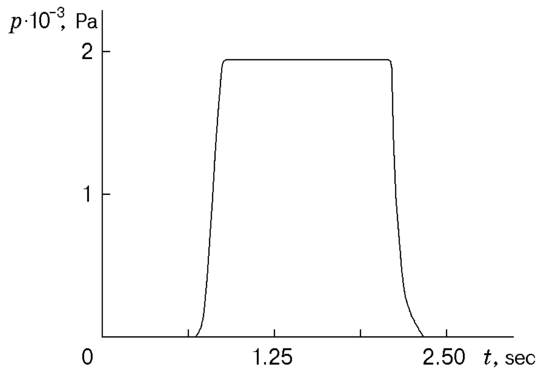


Fig. 2

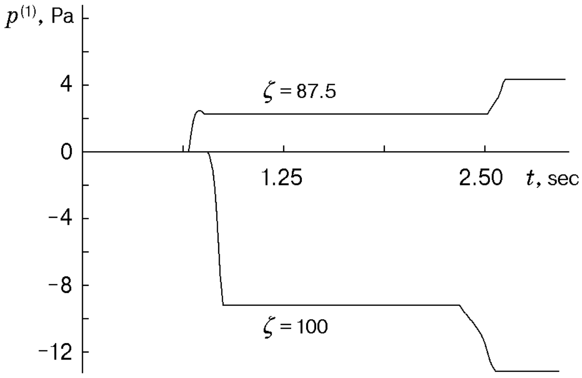


Fig. 3

The differential operators in the equations for $w^{(0)}$, $w^{(1)}$, and $w^{(2)}$ coincide with the operator of Eqs. (6)–(8) in [12], and the right side differs only in the term corresponding to the force $\Phi_t(v_{s0})$. Here we do not present these equations because they are too cumbersome.

The initial-boundary-value problem for the zeroth approximation for $\varepsilon$ was solved numerically. We employed an explicit three-layer finite-difference scheme for calculation of $u^{(0)}$ and $w^{(0)}$ and the method of characteristics for calculation of $v_{s0}$ and $p_0$. For the latter, we chose the invariants

$$I_1 = v_{s0} + a^2 p_0, \qquad I_2 = v_{s0} - a^2 p_0,$$

which reduce system (12) to canonical form.

For model calculations, we took parameters that correspond to a water flow in a steel pipe with an inner radius of 0.375 m, a length of 3011.25 m and a wall thickness of 0.01 m. The characteristic scales of the system are $l = 15$ m and $\omega = 100$ sec$^{-1}$. The profile shape is specified by the function

$$y(x) = \frac{4}{1 + 0.0025(x - 100)^2},$$

where $x$ and $y$ are measured in $l$. The shape of pressure oscillations at the entrance is specified by

$$F_0(\tau) = P_0\Big(1 - \cos\Big(2\pi\,\frac{\omega_0}{\omega}\,\tau\Big)\Big),$$

where $P_0$ is the constant amplitude of the signal.

The calculations show that in the zeroth approximation for $\varepsilon$, a pressure pulse is formed at a certain distance from the entrance (Fig. 2). Such a pulse was obtained in [1] and numerically determined in [15] but for different boundary conditions. The pulse height is equal to the oscillation amplitude $P_0$, i.e., the mean value of the pressure at the entrance.

In the first approximation for $\varepsilon$, the pressure is given by formulas (13). The quantities $w^{(1)}$ and $w^{(2)}$ are obtained from Eqs. (6)–(8) of [12], which are solved using three-layer finite-difference schemes.

It is shown that the main contribution to the pressure of the first approximation $p^{(1)}$ comes from the term

$$p^{(1)} \approx \frac{2rv_0}{a^2}\, f v_{s0},$$

as in the steady case [see (11)]. The physical meaning of this quantity and $p_0^{(1)}$ is the pressure difference on the length equal to the pipe diameter. The functions $p^{(1)}(t)$ for various values of $\zeta$ are shown in Fig. 3.

The calculation results for the displacement of the pipe wall and the fluid pressure in the zeroth approximation show good agreement between the mathematical model and known results.

Thus, formulas (4), (9), (10), (12), and (13) can be used to describe the motion of the system. In particular, one can obtain the dependence of the shape of pressure oscillations on the curvature of the pipeline axis. This dependence can be employed to design a system for controlling distortions of the pipeline profile.

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